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A LOWER BOUND FOR THE PERMANENT
OF A DOUBLY STOCHASTIC MATRIX

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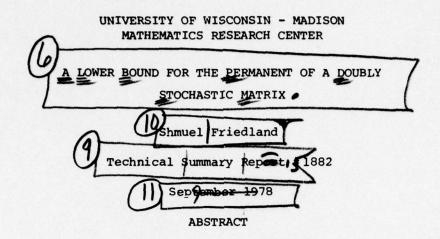**610 Walnut Street**
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ABSTRACT

It is shown here that the permanent of an $n \times n$ doubly stochastic matrix is not less than e^{-n}.

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SIGNIFICANCE AND EXPLANATION

Let A be a square matrix. The permanent p(A) of A is basically the determinant of A where all the summands appear with + signs. The notion p(A) arises naturally in many combinatorial settings where a count of the number of systems of distinct representatives of some configuration is required. In this paper we establish lower estimates of the right order for the permanents of doubly stochastic matrices. Recall that A is called a doubly stochastic matrix if all the entries of A are nonnegative and each row and column sum of A is equal to 1. Doubly stochastic matrices appear frequently in probability and combinatorics. The result established in this paper will have various applications in combinatorics and probability, in particular to Latin squares and block designs.

A LOWER BOUND FOR THE PERMANENT OF A DOUBLY
STOCHASTIC MATRIX

Shmuel Friedland[*]

1. Introduction.

Let A be an $n \times n$ matrix $(a_{ij})_1^n$. The permanent of A is defined by

$$(1.1) \qquad p(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \quad ,$$

where S_n is the symmetric group of order n. Let Ω_n be the set of all doubly stochastic matrices, that is, the set of all $n \times n$ matrices A satisfying

$$(1.2) \qquad a_{ij} \geq 0, \quad \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ij} = 1, \quad 1 \leq i,j \leq n \quad .$$

It is conjectured that any doubly stochastic matrix satisfies the inequality

$$(1.3) \qquad p(A) \geq n!/n^n \quad ,$$

with the equality holding only for the matrix J_n all of whose entries are 1/n. The problem of finding the minimum of p(A) on the set Ω_n goes back to van der Waerden [6]. In fact the inequality (1.3) is commonly referred as the van der Waerden conjecture. This conjecture is known to have applications to certain combinatorial problems. In this paper we establish the inequality

$$(1.4) \qquad p(A) \geq e^{-n}, \quad A \in \Omega_n \quad .$$

Recall that by Stirling's formula $n!/n^n \approx \sqrt{2\pi n} \, e^{-n}$. The previous lower bound known before was 1/n!. See [3] for the proof of this result and for the survey of the main results achieved in connection with the van der Waerden conjecture. Our starting point is the inequality due to T. Bang [1]

(1.5) $$p(A) \geq p(A \otimes J_m)^{1/m}, \quad m = 2^s ,$$

which holds for any non-negative A. Here by $B \otimes C$ we denote the tensor product of matrices B and C. Most of the paper is devoted to the proof of the equality

(1.6) $$\lim_{m \to \infty} p(A \otimes J_m)^{1/m} = e^{-n} ,$$

for any $n \times n$ doubly stochastic matrix A. Clearly in view of (1.5), the equality (1.6) implies the estimate (1.4).

2. Preliminary results.

Let $\sigma \in S_n$. Define

$$(2.1) \qquad P_\sigma = (\delta_{\sigma(i)j})_1^n \ .$$

By $\phi_{n,m}$ we denote the set of all matrices α with integer coefficients such that α/m is an $n \times n$ doubly stochastic matrix:

$$(2.2) \qquad \phi_{n,m} = \{\alpha | \alpha = (\alpha_{ij})_1^n, \ \alpha_{ij} \in \mathbb{Z}, \ \alpha/m \in \Omega_n\} \ .$$

From the classical Birkhoff theorem (e.g. [4]) it follows that

$$(2.3) \qquad \alpha = \sum_{k=1}^m P_{\sigma_k}, \ \alpha \in \phi_{n,m} \ .$$

Following T. Bang [1] we bring a formula for the permanent of $A \oplus J_m$.

Theorem 2.1. Let A be an $n \times n$ matrix. Then

$$(2.4) \qquad p(A \oplus J_m) = \frac{(m!)^{2n}}{m^{mn}} \sum_{\alpha \in \phi_{n,m}} \prod_{i,j=1}^n a_{ij}^{\alpha_{ij}}/\alpha_{ij}! \ .$$

Proof. Consider a term in $p(A \oplus J_m)$. It is of the form

$$(2.5) \qquad m^{-nm} \prod_{i,j=1}^n a_{ij}^{\alpha_{ij}} \ .$$

Clearly, each α_{ij} is a non-negative integer. Consider the rows $i, m+i,\ldots,(n-1)m+i$. These rows contribute to the product (2.5) the term $m^{-n} \prod_{j=1}^n a_{ij}^{\alpha_{ij}}$. Therefore, $\sum_{j=1}^n \alpha_{ij} = m$. In the same way one shows that $\sum_{i=1}^n \alpha_{ij} = m$. Thus, $(\alpha_{ij})_1^n \in \phi_{n,m}$. Vice versa, suppose that $\alpha \in \phi_{n,m}$. Then (2.3) holds. Let us view $A \oplus J_m$ as a block matrix

$$(2.6) \qquad A \oplus J_m = (A_{ij})_1^m, \ A_{ij} = m^{-1}A, \ i,j = 1,\ldots,m \ .$$

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In the block A_{rr} take a product $m^{-n} \prod_{i=1}^{n} a_{i\sigma_r(i)}$. By multiplying all these terms to-gether one obtains the expression

(2.7) $$m^{-nm} \prod_{r=1}^{m} \prod_{i=1}^{n} a_{i\sigma_r(i)} \ .$$

In view of (2.3) the expression (2.7) is equal to (2.5). We now compute the coefficient of the term (2.5). That is we are looking for the number of different ways to pick up α_{11} ele-ments $a_{11}/m,\ldots,\alpha_{nn}$ elements a_{nn}/m from the matrix $A \otimes J_m$ such that any two elements are not on the same row or column. We call such a choice an admissable choice. Let us label α_{ij} elements a_{ij}/m by $a_{ij}^{(1)}/m,\ldots,a_{ij}^{(\alpha_{ij})}/m$. Assume that for $\alpha_{ij} \geq 1$ the element $a_{ij}^{(k)}/m$ $(1 \leq k \leq \alpha_{ij})$ sits in the row $i + \mu(k,i,j)n$ and the column $j + \nu(k,i,j)n$ in the matrix $A \otimes J_m$. We assume that no two elements are placed in the same position. Let us call such a choice of positions a configuration. Two configurations considered to be equal if for each $\alpha_{ij} \geq 1$ and $1 \leq k \leq \alpha_{ij}$ the positions of the elements $a_{ij}^{(k)}/m$ coincide. Given a con-figuration one obtains a distinct confirguration by interchanging rows (columns) i and j, where $i \equiv j \pmod{n}$. It is easy to see that one can obtain any admissable configuration from a given configuration by interchanging appropriate rows and columns. Obviously, the rows (columns) $i, i+n,\ldots, i + (m-1)n$ $(1 \leq i \leq n)$ can be interchanged in $m!$ ways. Thus accord-ing to what we proved one has $(m!)^{2n}$ distinct configurations. Let us go back to the problem of determining the number of different ways to pick up the α_{ij} elements a_{ij}/m for $i,j = 1,\ldots,m$ (an admissable choice). Clearly any configuration gives rise to an admissable choice. We claim that to any admissable choice correspond $\prod_{i,j=1}^{n} \alpha_{ij}!$ distinct configuration. Indeed, for this choice, we have $\alpha_{ij}(\geq 1)$ places occupied by a_{ij}/m. In these α_{ij} places we put $a_{ij}^{(1)}/m,\ldots,a_{ij}^{(\alpha_{ij})}/m$. This can be done in $\alpha_{ij}!$ ways. Thus, to the given admissable choice correspond $\prod_{i,j=1}^{n} \alpha_{ij}!$ distinct configurations. Obviously, to two distinct admissable choices correspond distinct configurations. This demonstrates that the number of different ways to pick up α_{ij} elements a_{ij}/m, $i,j = 1,\ldots,m$, from the matrix $A \otimes J_m$, such that any two elements are not on the same row or column, is equal to $(m!)^{2n}(\prod_{i,j=1}^{n} \alpha_{ij}!)^{-1}$. The

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proof of the theorem is completed.

The permanent of a non-negative matrix can be estimated in terms of the permanent of $A \otimes J_m$ [1].

Theorem 2.2. Let s be a positive integer and $m = 2^s$. Then for any non-negative square matrix the following inequality holds

$$(2.8) \qquad p(A) \geq [p(A \otimes J_m)]^{1/m} \ .$$

Proof. We prove first the inequality (2.8) for $m = 2$. From (2.4) it follows

$$(2.9) \qquad p(A \otimes J_2) = \sum_{\alpha \in \Phi_{n,2}} \prod_{i,j=1}^{n} a_{ij}^{\alpha_{ij}} / \alpha_{ij}! \ .$$

On the other hand

$$(2.10) \qquad p(A)^2 = \sum_{\sigma, \eta \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \prod_{j=1}^{n} a_{j\eta(j)} \ .$$

Let $\alpha \in \Phi_{n,2}$. According to (2.3) $\alpha = p_\sigma + p_\eta$. Thus

$$(2.11) \qquad \prod_{i=1}^{n} a_{i\sigma(i)} \prod_{j=1}^{n} a_{j\eta(j)} = \prod_{i,j=1}^{n} a_{ij}^{\alpha_{ij}} \ .$$

Therefore, the coefficient of the term $\prod_{i,j=1}^{n} a_{ij}^{\alpha_{ij}}$ in $p(A)^2$ is a positive integer. The coefficient of this term in $p(A \otimes J_2)$ never exceeds 1. This establishes the inequality (2.8) for $m = 2$. Now the general case easily follows by induction. Indeed, suppose that (2.8) holds. Then

$$(2.12) \qquad p((A \otimes J_m) \times J_2)^{1/2m} \leq p(A \otimes J_m)^{1/m} \leq p(A) \ .$$

It is left to note that the tensor product is associative and

$$(2.13) \qquad J_m \otimes J_2 = J_{2m} \ .$$

End of proof.

Let A be an $n \times n$ doubly stochastic matrix. Suppose that the van der Waerden

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conjecture holds. Then

$$(2.14) \qquad p(A \otimes J_m)^{1/m} \geq [(nm)!/(nm)^{nm}]^{1/m} \ .$$

In what follows we estimate $p(A \otimes J_m)$ from above.

Lemma 2.1. __Let__ $A = (a_{ij})_1^n$ __be a__ __non-negative__ __matrix__. __Then__

$$(2.15) \qquad p(A \otimes J_m) \leq \frac{(m!)^n}{m^{mn}} \prod_{i=1}^n (\sum_{j=1}^n a_{ij})^m \ .$$

Proof. Consider the expression

$$(2.16) \qquad (\sum_{j=1}^n a_{ij})^m = m! \sum_{\beta_1 + \ldots + \beta_n = m} \prod_{j=1}^n a_{ij}^{\beta_j}/\beta_j! \ .$$

Choosing $\beta_j = \alpha_{ij}$, $j = 1, \ldots, n$, we get that the coefficient of the term $\prod_{j=1}^n a_{ij}^{\alpha_{ij}}$ in the expansion (2.16) is $m!/ \prod_{j=1}^n \alpha_{ij}!$. Expanding the expression

$$(2.17) \qquad \frac{(m!)^n}{m^{mn}} \prod_{i=1}^n (\sum_{j=1}^n a_{ij})^m$$

we see that the term $\prod_{i,j=1}^n a_{ij}^{\alpha_{ij}}$, $\alpha \in \Phi_{n,m}$, appears in (2.17) with the coefficient $\frac{(m!)^{2n}}{m^{mn}} \prod_{i,j=1}^n (\alpha_{ij}!)^{-1}$. As the expansion of the term (2.17) contains only non-negative terms from the identity (2.4) we deduce the inequality (2.15).

Thus, if A is a stochastic matrix the inequality (2.15) implies

$$(2.18) \qquad p(A \otimes J_m)^{1/m} \leq [(m!)^n/m^{mn}]^{1/m} \ .$$

Note that if $A = P_\sigma$ then the equality sign holds in (2.18). Recall the well known Stirling formula (e.g. [2, p.52])

$$(2.19) \qquad n! = \sqrt{2\pi n} \ n^n e^{-n} e^{\theta_n/12n}, \quad 0 < \theta_n < 1 \ .$$

Thus, if the van der Waerden conjecture holds by combining (2.14) and (2.18) with the Stirling formula we obtain

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(2.20)
$$\lim_{m \to \infty} p(A \otimes J_m)^{1/m} = e^{-n}$$

for any doubly stochastic matrix A. We shall prove the equality above in the next section, without assuming the validity of the van der Waerden conjecture. Of course, once (2.20) is established in view of Theorem 2.2 one deduces the inequality

(2.21)
$$p(A) \geq e^{-n}, \quad A \in \Omega_n .$$

Vice versa, (2.21) implies

(2.22)
$$p(A \otimes J_m)^{1/m} \geq e^{-n} .$$

Combining (2.22) with (2.18) we obtain again the equality (2.20). This manifests the equivalence of the relations (2.20) and (2.21).

3. <u>The main result</u>.

<u>Lemma 3.1</u>. <u>Let</u> n <u>and</u> r <u>be</u> <u>fixed</u> <u>positive</u> <u>integers</u>. <u>Then</u> <u>there</u> <u>exists</u> <u>two</u> <u>sequences</u> <u>of</u>
<u>positive</u> <u>numbers</u> $\{\epsilon_m\}_1^\infty$, $\{\delta_m\}_1^\infty$ <u>tending</u> <u>to</u> <u>zero</u> <u>such</u> <u>that</u> <u>for</u> <u>any</u> <u>non-negative</u> r <u>integers</u>
<u>satisfying</u> <u>the</u> <u>equality</u>

(3.1) $$k_1 + \ldots + k_r = m$$

<u>the</u> <u>following</u> <u>inequality</u> <u>holds</u>

(3.2) $$n^{nm}(1 + \delta_m)^{nm} \geq \frac{(nk_1)! \ldots (nk_r)!}{(k!)^n \ldots (k_r!)^n} \geq n^{nm}(1 - \epsilon_m)^{nm}, \quad (\epsilon_m < 1) \quad .$$

<u>Proof</u>. According to Stirling's formula (2.19)

$$\sqrt{2\pi} \, e \, j^{j+1/2}/e^j \geq j! \geq \sqrt{2\pi} \, j^{j+1/2}/e^j$$

for any $j \geq 1$. Without loss of generality we may assume that

$$1 \leq k_i, \quad i = 1, \ldots, s, \quad k_i = 0, \quad i = s+1, \ldots, r \quad .$$

Thus

$$\frac{(nk_1)! \ldots (nk_r)!}{(k_1!)^n \ldots (k_r!)^n} = \frac{(nk_1)! \ldots (nk_s)!}{(k_1!)^n \ldots (k_s!)^n} \geq \frac{(nk_1)^{nk_1+1/2} \ldots (nk_s)^{nk_s+1/2}}{((2\pi)^{(n-1)/2}e^n)^s (k_1)^{n(k_1+1/2)} \ldots (k_s)^{n(k_s+1/2)}} =$$

(3.3)

$$\frac{n^{mn+s/2}}{e^{ns}[(2\pi)^s k_1 \ldots k_s]^{(n-1)/2}} \geq \frac{n^{mn+s/2}}{[e^n (2\pi m/s)^{(n-1)/2}]^s} \quad .$$

The last part of the above inequality follows from the obvious fact

$$k_1 \ldots k_s \leq [(k_1 + \ldots + k_s)/s]^s = (m/s)^s \quad .$$

We also have

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$$\frac{(nk_1)!\ldots(nk_r)!}{(k_1!)^n\ldots(k_r!)^n} \le \frac{e^s (nk_1)^{nk_1+1/2}\ldots(nk_s)^{nk_s+1/2}}{(2\Pi)^{(n-1)s/2}(k_1)^{n(k_1+1/2)}\ldots(k_s)^{n(k_s+1/2)}} =$$

(3.4)

$$\frac{e^s n^{mn+s/2}}{[(2\Pi)^s k_1\ldots k_s]^{(n-1)/2}} \le \frac{e^s n^{mn+s/2}}{[(2\Pi)^s(m-s+1)]^{(n-1)/2}} \ .$$

Here we used the inequality

$$k_1\ldots k_s \ge (m-s+1)$$

since $k_1,\ldots,k_s$ are positive integers which sum up to m. Clearly the relations

(3.5)
$$\lim_{m\to\infty} \frac{n^{1+s/2nm}}{[e^n(2\Pi m/s)^{(n-1)/2}]^{s/nm}} = \lim_{m\to\infty} \frac{e^{s/nm}n^{1+s/2nm}}{[(2\Pi)^s(m-s+1)]^{(n-1)/2nm}} = n$$

prove the lemma.

Lemma 3.2. Let $p_1,\ldots,p_r$ be non-negative integers such that

(3.6)
$$p_1+\ldots+p_r = nm \ .$$

Then there exist non-negative integers $q_1,\ldots q_r$ with the following properties

(3.7)
$$q_1+\ldots+q_r = m \ ,$$

(3.8)
$$|p_j - nq_j| < n, \quad j = 1,\ldots,r$$

and

(3.9)
$$\frac{(nm)!}{p_1!\ldots p_r!} \le \left(\frac{n}{[\frac{n}{2}]}\right)^{r-1} \frac{(nm)!}{(nq_1)!\ldots(nq_r)!} \ .$$

Proof. By rearranging the indices we may assure that

(3.10)
$$p_j \not\equiv 0 \pmod{n}, \quad j = 1,\ldots,k \ ,$$

$$p_j \equiv 0 \pmod{n}, \quad j = k+1,\ldots,r \ .$$

In case that $k = 0$ (3.9) is trivial. Assume that $k \ge 2$. Without loss of generality we

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assume that

(3.11)
$$p_1 \le p_2 \quad .$$

Let

(3.12)
$$p_1 = nq_1 + t_1, \quad p_2 = n(q_1+1) - t_2, \quad 1 \le t_1, \quad t_2 < n \quad .$$

Suppose that

(3.13)
$$t_1 \le t_2 \quad .$$

Then

(3.14)
$$\frac{p_1! \, p_2!}{(nq_1)! \, (p_2+t_1)!} = \frac{(nq_1+1)\ldots(nq_1+t_1)}{[n(q_1+1)-t_2+1]\ldots[n(q_1+1)-(t_2-t_1)]} \quad .$$

Noting that the function $(nx+a)/(nx+b)$ is increasing on $(0,\infty)$ if $0 \le a \le b$ we obtain

(3.15)
$$\frac{p_1! \, p_2!}{(nq_1)! \, (p_2+t_1)!} \ge \frac{t_1!}{(n-t_2+1)\ldots[n-(t_2-t_1)]} \ge 1 \Big/ \binom{n}{t_1} \quad .$$

Recalling the well known inequality

$$\binom{n}{t_1} \le \binom{n}{[\frac{n}{2}]}$$

one deduces

(3.16)
$$1/(p_1! \, p_2!) \le \binom{n}{[\frac{n}{2}]} \Big/ [(nq_1)! \, (p_2+t_1)!] \quad .$$

Let

(3.17)
$$p'_j = p_{j+2}, \quad j = 1,\ldots,k-2, \quad p'_{k-1} = p_2+t_1, \quad p'_k = nq_1, \quad p'_j = p_j, \quad j = k+1,\ldots,r \quad .$$

According (3.16) we proved

(3.18)
$$\frac{(nm)!}{p_1!\ldots p_r!} \le \binom{n}{[\frac{n}{2}]} \frac{(nm)!}{p'_1!\ldots p'_r!} \quad .$$

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In case that $t_2 < t_1$, we replace p'_{k-1} and p'_k given by (3.17) by p_1-t_2 and $n(q_1+1)$ respectively and (3.18) is still valid. Note that $p'_1+\ldots+p'_r = nm$ and at most $k-1$ numbers out of $p'_1,\ldots,p'_r$ are not divisible by n. Continuing the procedure above we obtain the inequality (3.9).

Lemma 3.3. Let

(3.19) $$0 < \alpha_r \leq \alpha_{r-1} \leq \ldots \leq \alpha_1 \leq 1 \ .$$

Then

(3.20) $$\left(\sum_{j=1}^{r} \alpha_j \right)^{nm} \leq \alpha_r^{-(n-1)r} \binom{n}{[\frac{n}{2}]}^{r-1} (2n-1)^r \times$$

$$\sum_{k_1+\ldots+k_r=m} \frac{(nm)!}{(nk_1)!\ldots(nk_r)!} \alpha_1^{nk_1} \ldots \alpha_r^{nk_r} \ .$$

Proof. Recall that

(3.21) $$\left(\sum_{j=1}^{r} \alpha_j \right)^{nm} = \sum_{p_1+\ldots+p_r=nm} \frac{(nm)!}{p_1!\ldots p_r!} \alpha_1^{p_1} \ldots \alpha_r^{p_r} \ .$$

Consider a term

(3.22) $$\frac{(nm)!}{p_1!\ldots p_r!} \alpha_1^{p_1} \ldots \alpha_r^{p_r} \ .$$

According to Lemma 3.2 there exist positive integers $q_1,\ldots,q_r$ such that (3.7) – (3.9) hold. As $\alpha_j \leq 1$ we also have

(3.23) $$\alpha_1^{p_1} \ldots \alpha_r^{p_r} \leq (\alpha_1^{-(n-1)} \alpha_1^{nq_1}) \ldots (\alpha_r^{-(n-1)} \alpha_r^{nq_r}) \leq \alpha_r^{-(n-1)r} (\alpha_1^{nq_1} \ldots \alpha_r^{nq_r}) \ .$$

So

(3.24) $$\frac{(nm)!}{p_1!\ldots p_r!} \alpha_1^{p_1} \ldots \alpha_r^{p_r} \leq \binom{n}{[\frac{n}{2}]}^{r-1} \frac{(nm)!}{(nq_1)!\ldots(nq_r)!} \alpha_r^{-(n-1)r} \alpha_1^{nq_1} \ldots \alpha_r^{nq_r} \ .$$

For a fixed r integers $q_1,\ldots q_r$ we can have at most $(2n-1)^r$ types of r integers

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$P_1, \ldots, P_r$ satisfying (3.8). Thus, by using the inequality (3.24) for each summand appearing in (3.21) we establish the lemma.

We are now ready to prove our main result.

Theorem 3.1. Let A be an $n \times n$ doubly stochastic matrix. Then

$$(3.25) \qquad \lim_{m \to \infty} p(A \otimes J_m)^{1/m} = e^{-n} \ .$$

Proof. From the classical Birkhoff theorem (e.g. [4]) it follows that

$$(3.26) \qquad A = \sum_{j=1}^{r} \alpha_j P_{\sigma_j}, \quad \alpha_j > 0, \quad j = 1, \ldots, r, \quad \sum_{j=1}^{r} \alpha_j = 1 \ .$$

Without loss of generality one may assume that $\{\alpha_j\}_1^r$ is a decreasing sequence, i.e. (3.19) holds. We claim that

$$(3.27) \qquad p(A \otimes J_m) \geq \frac{(m!)^{2n}}{m^{mn}} \sum_{k_1 + \ldots + k_r = m} \frac{\alpha_1^{nk_1} \ldots \alpha_r^{nk_r}}{(k_1!)^n \ldots (k_r!)^n} \ .$$

Indeed, let $k_1, \ldots, k_r$ be r non-negative integers which sum up to m. Consider

$$(3.28) \qquad \beta = \sum_{\ell=1}^{r} k_\ell P_{\sigma_\ell} \in \Phi_{n,m} \ .$$

From the expansion (2.4) for the permanent of $p(A \otimes J_m)$ it follows that one has a term of the form

$$(3.29) \qquad \frac{(m!)^{2n}}{m^{mn}} \prod_{i,j=1}^{r} a_{ij}^{\beta_{ij}} / \beta_{ij}! \ .$$

Recall that $P_\sigma = (\delta_{\sigma(i)j})_1^n$. So

$$(3.30) \qquad \beta_{ij} = \sum_{\ell=1}^{r} k_\ell \delta_{\sigma_\ell(i)j} \ ,$$

$$(3.31) \qquad a_{ij} = \sum_{s=1}^{r} \alpha_s \delta_{\sigma_s(i)j} \ .$$

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Thus, the multinomial expansion of $a_{ij}^{\beta_{ij}}$ contains the term

$$(3.32) \qquad (\beta_{ij})! \; \frac{\alpha_1^{k_1 \delta_{\sigma_1(i)j}} \cdots \alpha_r^{k_r \delta_{\sigma_r(i)j}}}{(k_1 \delta_{\sigma_1(i)j})! \cdots (k_r \delta_{\sigma_r(i)j})!} \; .$$

This implies at once that the expansion of (3.29) contains a term of the form

$$(3.33) \qquad \frac{(m!)^{2n}}{m^{mn}} \prod_{i,j=1}^{n} \frac{\alpha_1^{k_1 \delta_{\sigma_1(i)j}} \cdots \alpha_r^{k_r \delta_{\sigma_r(i)j}}}{(k_1 \delta_{\sigma_1(i)j})! \cdots (k_r \delta_{\sigma_r(i)j})!} = \frac{(m!)^{2n}}{m^{mn}} \frac{\alpha_1^{nk_1} \cdots \alpha_r^{nk_r}}{(k_1!)^n \cdots (k_r!)^n} \; .$$

As the multinomial expansion of $a_{ij}^{\beta_{ij}}$ contains only positive summands from (2.4) and (3.33) we obtain the inequality (3.27).

Consider the identity

$$(3.34) \qquad \frac{\alpha_1^{nk_1} \cdots \alpha_r^{nk_r}}{(k_1!)^n \cdots (k_r!)^n} = \frac{(nk_1)! \cdots (nk_r)!}{(k_1!)^n \cdots (k_r!)} \; \frac{\alpha_1^{nk_1} \cdots \alpha_r^{nk_r}}{(nk_1)! \cdots (nk_r)!} \; .$$

According to Lemma 3.1

$$(3.35) \qquad \frac{\alpha_1^{nk_1} \cdots \alpha_r^{nk_r}}{(k_1!)^n \cdots (k_r!)^n} \geq n^{nm}(1-\epsilon_m)^{nm} \frac{\alpha_1^{nk_1} \cdots \alpha_r^{nk_r}}{(nk_1)! \cdots (nk_r)!} \; .$$

From (3.27) and (3.35) it follows

$$(3.36) \qquad p(A \otimes J_m) \geq \frac{(m!)^{2n} \, n^{nm}(1-\epsilon_m)^{nm}}{m^{mn} \, (mn)!} \sum_{k_1 + \ldots + k_r = m} (nm)! \; \frac{\alpha_1^{nk_1} \cdots \alpha_r^{nk_r}}{(nk_1)! \cdots (nk_r)!} \; .$$

Applying the inequality (3.20) and noting that $\sum_{j=1}^{r} \alpha_j = 1$ we finally deduce

$$(3.37) \qquad p(A \otimes J_m) \geq \frac{(m!)^{2n} \, n^{nm}(1-\epsilon_m)^{nm}}{m^{mn} \, (mn)!} \alpha_r^{(n-1)r} \binom{n}{[\frac{n}{2}]}^{-(r-1)} (2n-1)^{-r} \; .$$

Using Stirling's formula and the fact that $\lim_{m \to \infty} \epsilon_m = 0$ we get

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(3.38) $$\lim_{m \to \infty} \inf p(A \otimes J_m)^{1/m} \geq e^{-n} \ .$$

On the other hand the inequality (2.18) implies

(3.39) $$\lim_{m \to \infty} \sup p(A \otimes J_m)^{1/m} \leq e^{-n} \ .$$

The above two inequalities establish the theorem.

Combine Theorems 2.2 and 3.1 to deduce

Theorem 3.2. Let A be an n × n doubly stochastic matrix. Then

(3.40) $$p(A) \geq e^{-n} \ .$$

We conclude our paper by an application of the inequality (3.40) to the problem of Marshall Hall (unpublished). The problem is to estimate from below the permanent of 0 - 1 matrix having exactly three 1 in each row and column. In what follows we consider a larger class of matrices.

Corollary 3.1. Let α be an n × n matrix which is a sum of three permutation matrices. That is α belongs to the set $\phi_{n,3}$. Then

(3.41) $$p(\alpha) \geq \left(\frac{3}{e}\right)^n \ .$$

A lower bound known before was n [5].

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It is shown that the permanent of an $n \times n$ doubly stochastic matrix is not less than e^{-n}.

e to the -n power